

Constacyclic Codes Over Finite Principal Ideal Rings

Aicha Batoul, Kenza Guenda and T. Aaron Gulliver *

Abstract

In this paper, we give an important isomorphism between constacyclic codes and cyclic codes over finite principal ideal rings. Necessary and sufficient conditions for the existence of non-trivial cyclic self-dual codes over finite principal ideal rings are given.

1 Introduction

Although codes over rings are not new [?], they have attracted significant attention from the scientific community only since 1994, when Hammons et al. [25] established a fundamental connection between non-linear binary codes and linear codes over \mathbb{Z}_4 . In [25], it was proven that some of the best non-linear codes, such as the Kerdock, Preparata, and Goethal codes can be viewed as linear codes over \mathbb{Z}_4 via the Gray map from \mathbb{Z}_4^n to \mathbb{F}_2^{2n} . The link between self-dual codes and unimodular lattices was given by Bonnecaze et al. [8] and Bannai et al. [5]. These results created a great deal of interest in self-dual codes over a variety of rings, see [34] and the references therein. Dougherty et al. [16, 20] used the Chinese remainder theorem to generalize the structure of codes over principal ideal rings. They gave conditions on the existence of self-dual codes over principal ideal rings in [20].

Dougherty [?] recently posed a number of problems concerning codes over rings. Several of these are answered in this paper. In particular, we give necessary and sufficient conditions on the existence of self-dual codes over principal ideal rings. The existence of such codes requires the existence of self-dual codes over all the base finite chain ring. We also give the structure of constacyclic codes over finite principal ideal rings. The projection and the lift of these codes is described using a generalization of the Hensel Lift Lemma and the structure of the ideals of $R[x]/\langle x^n - \lambda \rangle$. Finally, infinite families of self-dual codes are given over principal

*A. Batoul and K. Guenda are with the Faculty of Mathematics USTHB, University of Science and Technology of Algiers, Algeria. T. Aaron Gulliver is with the Department of Electrical and Computer Engineering, University of Victoria, PO Box 3055, STN CSC, Victoria, BC, Canada V8W 3P6. email: agullive@ece.uvic.ca.

ideal rings. Codes over rings are a generalization of codes over fields. In [25], it was proven that some well known non-linear codes, such as the Kerdock, Preparata, and Goethal codes, are the image of linear codes over \mathbb{Z}_4 via the Gray map from \mathbb{Z}_4^n to \mathbb{F}_2^{2n} . These results generated a great deal of interest in self-dual codes over a variety of rings, e.g. [22, 24, 34]. In addition to self-dual codes over rings being theoretically important, they also have many practical applications, for example they are related to unimodular lattices [5, 8]. Several researchers have considered self-dual codes over rings [13, 19, 22]. The structure of cyclic codes over \mathbb{Z}_{p^a} was first given by Calderbank and Sloane [10]. This motivated others (e.g. [1, 6, 36]), to investigate the structure of cyclic and negacyclic codes over chain rings. Kanwar, Dinh and López-Permouth [12, 27] generalized this structure to cyclic and negacyclic codes over finite chain rings, and considered the self-duality of these codes. More recently, the structure given in [12] has been generalized to constacyclic codes [24]. Motivated by an open question posed by Jia et al. [26] on the structure of cyclic self-dual codes over rings, we give in this paper necessary and sufficient conditions on the existence of non-trivial cyclic self-dual codes over finite chain rings. Another motivation of the present work is the characterization of those integers n for which $p^i \not\equiv -1 \pmod n$ for all i and p odd. This is required to determine the non-trivial cyclic self-dual codes given by Dinh and López-Permouth [12, p. 1734]. We prove that in the case of even nilpotency, there exists a non-trivial cyclic self-dual code of length n over a finite principal ideal ring R , such that the residual field has cardinality p^r , if and only if $\text{ord}_n(p^r)$ is odd. We also prove that there are no free cyclic self-dual codes over finite chain rings with odd characteristic. Furthermore, it is proven that a self-dual code over a chain ring cannot be the lift of a binary cyclic self-dual code. We give explicit expressions for the number of cyclic self-dual codes over chain rings and provide examples.

2 Preliminaries

Since principal ideal rings are Frobenius rings we need to give some tools necessities for the after.

3 Commutative Frobenius rings

We assume that all rings are commutative and with identity. For all unexplained terminology and more detailed we refer to [?] (related algebra) and to [30]. A finite commutative ring R is Frobenius if the R -module R is injective. Alternatively, we can say a finite commutative ring is Frobenius if $R/J(R)$ is isomorphic to $\text{Soc}(R)$, where $J(R)$ is the Jacobson radical of the ring R and $\text{Soc}(R)$ is the Socle of the ring. Recall that the Jacobson radical is the intersection of all maximal ideals in the ring and the Socle of the ring is the sum of the minimal R -submodules. Finite Frobenius rings are very important in coding theory for

several reason and precisely for the following equality [?]: A code C over a finite Frobenius ring R and its dual satisfy the following

$$|C||C^\perp| = |R|^n, \text{ and } (C^\perp)^\perp = C. \quad (1)$$

If I is an ideal of a finite ring, then the chain $I \supset I^2 \supset I^3 \supset \dots$ stabilizes. The smallest $e \geq 1$ such that $I^e = I^{e+1} = \dots$ is called the index of stability of I . If I is nilpotent, then the smallest $e \geq 1$ such that $I^e = 0$ is called the index of nilpotency of I and is the same as the index of stability of I . Note that if R is local, with maximal ideal M then we have necessarily $M^e = M^{e+1} = \dots = 0$. Thus in the case of finite local rings, the index of stability of M is in fact the index of nilpotency of M . On the other side, if R has at least two maximal ideals, then for any maximal ideal $J, J^e = J^{e+1} = \dots \neq 0$. Otherwise, if $I \neq J$ is another maximal ideal, we would have $I \supset (0) = J^e$, hence $J \subset I$, a contradiction.

Let R be a ring, I an ideal of R . Denote by $\Psi_i : R \longrightarrow R/I$ the canonical homomorphism $x \longmapsto x + I$. If n is a fixed positive integer we also denote $\Psi_i : R^n \longrightarrow (R/I)^n$ the canonical R -linear map

$$(x_1, \dots, x_n) \longmapsto (x_1 + I, \dots, x_n + I)$$

Let R be a finite ring,

Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$ the maximal ideals of R e_1, \dots, e_k their indices of stability. Then the ideals $\mathfrak{m}_1^{e_1}, \mathfrak{m}_2^{e_2}, \dots, \mathfrak{m}_k^{e_k}$ are relatively prime in pairs, and $\prod_{i=1}^k \mathfrak{m}_i^{e_i} = \cap_{i=1}^k \mathfrak{m}_i^{e_i} = \{0\}$. By the ring version of the Chinese Remainder Theorem, the canonical ring homomorphism

$$\Psi : R \longrightarrow \prod_{i=1}^k R/\mathfrak{m}_i^{e_i}$$

defined by $x \longmapsto (x + \mathfrak{m}_1^{e_1}, \dots, x + \mathfrak{m}_k^{e_k})$, is an isomorphism. Denote the local rings $R/\mathfrak{m}_i^{e_i}$ by R_i ($i = 1, \dots, k$). The maximal ideal of R_i has nilpotency index e_i . Note that R is Frobenius if and only if each R_i is Frobenius [?]. For a code $C \subset R^n$ over R and the maximal ideal \mathfrak{m}_i of R , the \mathfrak{m}_i -projection of C is defined by $C_i = \Psi_i(C)$ where $\Psi_i : R^n \longrightarrow R_i^n$ is the canonical map. We denote by $\Psi : R^n \longrightarrow \prod_{i=1}^k R_i^n$ the map defined by $\Psi(u) = (\Psi_1(u), \dots, \Psi_k(u))$ for $u \in R^n$. By the module version of the Chinese Remainder Theorem, the map Ψ is an R -module isomorphism and

$$C \simeq C_1 \times C_2 \times \dots \times C_k$$

Conversely, given codes C_i of length n over R_i ($i = 1, \dots, k$), we define the code $C = CRT(C_1, \dots, C_k)$ of length n over R in the following way

$$\begin{aligned} C &= \{\Psi^{-1}(u_1, \dots, u_k); u_i \in C_i (i = 1, \dots, k)\} \\ &= \{u \in R^n; \Psi_i(u) \in C_i (i = 1, \dots, k)\} \end{aligned}$$

then the code $C = CRT(C_1, \dots, C_k)$ is called the Chinese product of the code C_i .

As a particular case of the above discussion (and with the above notation) we have

Theorem 3.1 *Let R be a finite Frobenius ring, n a positive integer. Then*

$$R^n = CRT(R_1^n, R_2^n, \dots, R_k^n)$$

where R_i is local Frobenius ring.

Lemma 3.2 *Let C_1, C_2, \dots, C_k be codes of length n with C_i a code over R_i , and let $C = CRT(C_1, C_2, \dots, C_k)$ then*

$$(i) |C| = \prod_{i=1}^k |C_i|.$$

(ii) C is a free code if and only if each C_i is a free code of the same rank.

Notice that if two codes are free but not of the same rank then the cardinality of their image under CRT is not that of free code. For example, the Chinese product of free code of rank 1 over \mathbb{Z}_2 and a code of rank 2 over \mathbb{Z}_3 has cardinality $2^1 \times 3^2 = 18$ which is not 6^k for any integer k .

3.1 Finite Principal Ideal Rings

Lemma 3.3 (*[9], p. 54, Proposition 6*) *Let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ be ideals of R , relatively prime in pairs, and let $\mathfrak{a} = \cap_{i=1}^n \mathfrak{a}_i$. For every R -module M , the canonical homomorphism $M \rightarrow \prod_{i=1}^n (M/\mathfrak{a}_i M)$ is surjective and has kernel $\mathfrak{a}M$.*

Let \mathfrak{a}_i be an ideal of a ring R , and denote $R_i = R/\mathfrak{a}_i$. Hence we have a canonical epimorphism $\psi_i : R \rightarrow R_i$.

Proposition 3.4 *Let R be a finite commutative ring. Then the following are equivalent.*

(i) R is a principal ideal ring.

(ii) R is isomorphic to a finite product of chain rings.

Moreover, the decomposition in (ii) is unique up to the order of factors. It has the form $R \cong \prod_{i=1}^k R/\mathfrak{m}_i^{t_i}$, where $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$ are maximal ideals of R , and t_1, t_2, \dots, t_k are the corresponding indexes of stability.

If R is a finite principal ideal ring, we say that the decomposition of R into a product of finite chain rings, as in (ii), is a *canonical decomposition* of R . The ideal $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$ in this case is called a *direct decomposition* of R .

Lemma 3.5 (*[?], p 110, Proposition 10*) *Let R be a finite ring and $(\mathfrak{a}_i)_{i=1}^n$ be ideals of R . The following are equivalent:*

- i) The family $(\mathfrak{a}_i)_{i=1}^n$ is a direct decomposition of R .
- ii) For $i \neq j$, \mathfrak{a}_i and \mathfrak{a}_j are relatively prime and $\bigcap_{i=1}^n \mathfrak{a}_i = \{0\}$.
- iii) For $i \neq j$, \mathfrak{a}_i and \mathfrak{a}_j are relatively prime and $\prod_{i=1}^n \mathfrak{a}_i = \{0\}$.
- iv) There exists a family $(e_i)_{i=1}^n$ of idempotents of R such that $e_i e_j = 0$ for $i \neq j$, $1 = \sum e_i$ and $\mathfrak{a}_i = R(1 - e_i)$ for $i = 1, \dots, n$.

Proposition 3.6 ([16], Proposition 2.4) *With the notation as above, $(\mathfrak{a}_i)_{i=1}^n$ a direct decomposition of R and M an R -module.:*

- i) For each $i \in \{1, \dots, n\}$ the submodule $M_i = e_i M$ is a complement of the submodule $\mathfrak{a}_i M = (1 - e_i)M$ and so the map

$$\psi_i : M_i \longrightarrow M/\mathfrak{a}_i M, x \longmapsto x + \mathfrak{a}_i M$$

- ii) The action of R on M , $(r, x) \longmapsto rx$ can be identified with the componentwise actions $((r_1 + \mathfrak{a}_1, \dots, r_k + \mathfrak{a}_k), x_1 \oplus x_2 \oplus \dots \oplus x_n) \longmapsto r_1 x_1 \oplus \dots \oplus r_n x_n$ $((r_1 + \mathfrak{a}_1, \dots, r_k + \mathfrak{a}_k), (x_1 + \mathfrak{a}_1 M, \dots, x_n + \mathfrak{a}_n M)) \longmapsto r_1 x_1 + \mathfrak{a}_1 M, \dots, r_n x_n + \mathfrak{a}_n M$ of $\prod_{i=1}^n R/\mathfrak{a}_i$ on $M = \bigoplus_{i=1}^n M_i$ and $\prod_{i=1}^n M/\mathfrak{a}_i M$ respectively
- iii) Every sub-module N of M is an internal direct sum of submodule $N_i = e_i N \subset M_i$ which are isomorphic via ψ_i with the submodule $N'_i = (\mathfrak{a}_i M + e_i N)/\mathfrak{a}_i M$ of $M/\mathfrak{a}_i M$ for $i = 1, 2, \dots, n$. Each N'_i is isomorphic to $N/\mathfrak{a}_i N$ and so the decomposition $N \longrightarrow \bigoplus_{i=1}^n N'_i \subset \bigoplus_{i=1}^n M/\mathfrak{a}_i M$ canonically corresponds to the decomposition $N \longrightarrow \bigoplus_{i=1}^n N/\mathfrak{a}_i N$. Conversely, if for every $i = 1, \dots, n$ N'_i is a submodule of $M/\mathfrak{a}_i M$ then there is a unique submodule $N = \bigoplus_{i=1}^n N_i$ of M such that N is isomorphic with $\bigoplus_{i=1}^n N'_i$ via $\psi = \bigoplus_{i=1}^n \psi_i$

Let R be a finite ring. A code is a subset of R^n and linear code over R is an R -submodule of R^n . In this case we say the code has length n . We attach the standard inner product to the ambient space, i.e., $[u, v] = \sum u_i v_i$. The dual code C^\perp of C is defined by

$$C^\perp = \{u \in R^n \mid [u, v] = 0 \text{ for all } v \in C\}. \quad (2)$$

We say that a code is self-orthogonal if $C \subseteq C^\perp$, and self-dual if $C = C^\perp$. The Hamming weight of a vector from R^n is the number of non-zero coordinates in that vector and the minimum weight is the smallest of all non-zero weights in a code. A code $C \subset R^n$ is called a free code if C is a free R -module, that C is isomorphic to the R -module R^k for some k .

We refer to C as the *Chinese product of codes* C_1, C_2, \dots, C_k [21].

3.2 Finite Chain Rings

In this subsection, we summarize the necessary results from ([24] [12] [33]). A finite *chain ring* is a finite commutative ring R with $1 \neq 0$ such that its ideals are ordered by inclusion. The ring R is called a local ring if R has a unique maximal ideal. A finite commutative ring is a finite chain ring if and only if it is a local principal ideal ring [12, Proposition 2.1]. Let \mathfrak{m} be the maximal ideal of the finite chain ring R . Since R is principal, there exists a generator $\gamma \in R$ of \mathfrak{m} . Then γ is nilpotent with nilpotency index some integer e . The ideals of R form the following chain

$$\langle 0 \rangle = \langle \gamma^e \rangle \subsetneq \langle \gamma^{e-1} \rangle \subsetneq \dots \subsetneq \langle \gamma \rangle \subsetneq R.$$

The nilradical of R is then $\langle \gamma \rangle$, so all the elements of $\langle \gamma \rangle$ are nilpotent. Hence the elements of $R \setminus \langle \gamma \rangle$ are units. Since $\langle \gamma \rangle$ is a maximal ideal, the residue ring $\frac{R}{\langle \gamma \rangle}$ is a field which we denote by K . The natural surjective ring morphism is given by $(-)$ as follows

$$\begin{aligned} - : R &\longrightarrow K \\ a &\longmapsto \bar{a} = a \pmod{\gamma} \end{aligned} \tag{3}$$

Let $|R|$ denote the cardinality of R , and R^* the multiplicative group of all units in R . We also have that if $|K| = q = p^r$ for some integer r , then

$$|R| = |K| \cdot |\langle \gamma \rangle| = |K| \cdot |K|^{e-1} = |K|^e = p^{er}. \tag{4}$$

We define the characteristic of the finite chain ring as the prime number p which is the characteristic of the residue field K of R . Note that this is not the usual definition of the characteristic of a ring.

A code C and its dual satisfy the following

$$|C||C^\perp| = q^{en} = |R|^n, \text{ and } (C^\perp)^\perp = C. \tag{5}$$

Remark 3.7 From (5), there exists a self-dual code of length n over R if and only if en is even. This explains for example why there are no self-dual codes of odd length over \mathbb{Z}_8 [13]. If e is even, there exists a trivial self-dual code of length n given by the generator matrix $G = \gamma^{\frac{e}{2}} I_n$.

Let n be a positive integer and q a prime power. We denote by $\text{ord}_n(q)$ the multiplicative order of q modulo n , which is the smallest integer l such that $q^l \equiv 1 \pmod{n}$.

4 Constacyclic Codes over Finite Principal Ideal Rings

This section considers codes over finite commutative rings which are finite principal ideal. Let R be a commutative ring with unity. For a given unit $\lambda \in R$, a code C is said to be constacyclic, or more generally, λ constacyclic, if $(\lambda c_{n-1}, c_0, c_1, \dots, c_{n-2}) \in C$ whenever $(c_0, c_1, \dots, c_{n-1}) \in C$. For example, cyclic and negacyclic codes correspond to $\lambda = 1$ and -1 , respectively.

The main goal of this section is to prove the existence of an isomorphism between constacyclic codes and cyclic codes over finite principal ideal rings. This justifies our restriction to cyclic codes in the following sections.

But before we recall some result given in [3]

4.1 Constacyclic Codes over Finite chain Rings

Let R be a finite chain ring, with residue field \mathbb{F}_q .

Definition 4.1 *A polynomial f of $R[x]$ is called basic irreducible if \bar{f} is irreducible in $\bar{R}[x] = [x]$. Two polynomials f and g in $R[x]$ are called coprime if*

$$R[x] = \langle f \rangle + \langle g \rangle.$$

Let λ be a unit in a finite chain ring R . If a polynomial $f(x)$ divides $x^n - \lambda$, (say $x^n - \lambda = f(x)g(x)$), we refer to $g(x) = \frac{x^n - \lambda}{f(x)}$ as $\hat{f}(x)$.

Theorem 4.2 ([24, Theorem 4.7]) *Let λ be a unit in a finite chain ring R with characteristic p . When $(n, p) = 1$, the polynomial $x^n - \lambda$ factors uniquely as a product of monic basic irreducible pairwise coprime polynomials over R . Furthermore, there is a one-to-one correspondence between the set of basic irreducible polynomial divisors of $x^n - \lambda$ in $R[x]$ and the set of irreducible divisors of $\overline{x^n - \lambda}$ in K .*

Theorem 4.3 ([24, Theorem 4.16, Corollary 4.17]) *Let R be a finite chain ring and C a λ constacyclic code over $R[x]$ of length n such that $(n, p) = 1$, where p is the characteristic of \bar{R} . Then there exists a unique family of pairwise coprime polynomials F_0, \dots, F_e in $R[x]$ such that $F_0 \dots F_e = x^n - \lambda$ and $C = \langle \hat{F}_1, \gamma \hat{F}_2, \dots, \gamma^{e-1} \hat{F}_e \rangle$, where $\hat{F}_j = \frac{x^n - 1}{F_j}$ for $0 < j \leq e$. Moreover, we have that*

$$|C| = (K)^{\sum_{j=0}^{e-1} (e-j) \deg F_{e+1}}, \quad (6)$$

and the ring $R[x]/\langle x^n - \lambda \rangle$ is a principal ideal ring.

It was proven by Dinh and López-Permouth [12] that negacyclic codes of odd length are isomorphic to cyclic codes of the same length if $(n, p) = 1$. In the following, we give an isomorphism in more general case. For this, let λ, δ be units of R such that $\lambda = \delta^n$.

Proposition 4.4 [4] *Let n be an integer and λ, δ units such that $\lambda = \delta^n$. Let μ be the map $\mu : R[x]/\langle x^n - 1 \rangle \mapsto R[x]/\langle x^n - \lambda \rangle$ defined by $\mu(c(x)) = c(\delta^{-1}x)$. Then we have that μ is a ring isomorphism.*

From Theorem 4.3, we have that the ideals in $R[x]/\langle x^n - \lambda \rangle$ are principal ideals. Then the following result is a straightforward consequence of Proposition 4.4.

Corollary 4.5 [4] *Let R be a finite chain ring and λ, δ units in R such that $\lambda = \delta^n$. A subset I in $R[x]$ is an ideal in $R[x]/\langle x^n - 1 \rangle$ if and only if $\mu(I)$ is an ideal in $R[x]/\langle x^n - \lambda \rangle$. Equivalently, the set C is a cyclic code of length n over the chain ring R if and only if $\mu(C)$ is a λ -constacyclic code of length n over R .*

4.2 Constacyclic Codes over Finite Principal Ideal Rings

In the following we generalize the above result to finite principal ideal rings but before we need to recall some results about them.

For the after, we need the followings lemmas.

Lemma 4.6 [30] *Let R^* denote the group of units of a finite ring R , if R decomposes as a direct sum $R = R_1 \oplus \cdots \oplus R_k$ of rings R_i then R^* decomposes naturally as a direct product $R^* = R_1^* \oplus \cdots \oplus R_k^*$ of groups.*

So let $\lambda \in R^*$ a unit in R then $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where each $\lambda_i \in R_i^*$

Remark 4.7 *If $\prod R_{i=1}^k$ is a direct decomposition of a finite principal ideal ring R and $\lambda \in R^*$ then $\lambda = CRT(\lambda_1, \lambda_2, \dots, \lambda_k)$ where each $\lambda_i \in R_i^*$.*

Lemma 4.8 *Let R be a finite principal ideal rings and $\prod R_{i=1}^k$ its direct decomposition (ie $R = CRT(R_1, R_2, \dots, R_k)$).*

R has units λ and δ such that $\lambda = \delta^n$ if and only if each finite chain ring R_i has units λ_i and δ_i such that $\lambda_i = \delta_i^n$.

Proof. If there exist units $\lambda_i, \delta_i \in R_i$ such that $\lambda_i = \delta_i^n$ for $1 \leq i \leq k$. Then $\lambda = CRT(\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\delta = CRT(\delta_1, \delta_2, \dots, \delta_k)$ satisfies $\lambda = \delta^n$.

If R has units λ and δ such that $\lambda = \delta^n$ then $\lambda_i = \psi_i(\lambda) = \psi_i(\delta^n) = \psi_i(\delta)^n = \delta_i^n$ ■

Theorem 4.9 *Let R be a finite principal ideal ring, $\prod R_{i=1}^k$ its direct decomposition and λ be a unit in R such that $\lambda = CRT(\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_i \in R_i^*$. Let $C = CRT(C_1, C_2, \dots, C_k)$ be a code over R of length n with local components codes C_i of length n over R_i ; $1 \leq i \leq k$ be codes of length an integer n . Then C is λ -constacyclic code over R if and only if each C_i is λ_i -constacyclic code over R_i .*

Proof. For $i \in \{1, \dots, k\}$ let \mathbb{F}_{q_i} be the residual fields of R_i . Further, define the following

$$\begin{array}{ccc} \text{ring homomorphism } \phi: R[x]/\langle x^n - \lambda \rangle & \longrightarrow & R_i[x]/\langle x^n - \lambda_i \rangle \\ a_0 + a_1x + \dots + a_{n-1}x^{n-1} & \longmapsto & \psi_i(a_0) + \psi_i(a_1)x + \dots + \psi_i(a_{n-1})x^{n-1} \end{array}$$

Next define

$$\phi: R[x]/(x^n - \lambda) \longrightarrow R_1[x]/(x^n - \lambda_1) \times R_2[x]/(x^n - \lambda_2) \times \dots \times R_k[x]/(x^n - \lambda_k)$$

where

$$\phi(f(x)) = (\phi_1(f(x)), \phi_2(f(x)), \dots, \phi_k(f(x))).$$

If I is an ideal of $R[x]/(x^n - \lambda)$, then $\phi_i(I)$ is an ideal of $R_i[x]/(x^n - \lambda_i)$.

Conversely for ideals I_i in $R_i[x]/(x^n - \lambda_i)$ we define

$$\phi^{-1}(I_1, I_2, \dots, I_k).$$

Note that

$$I = CRT(I_1, I_2, \dots, I_k)$$

is the unique ideal in $R[x]/(x^n - \lambda)$ that is congruent to I_i in R_i . By the generalized Chinese Remainder Theorem this map is well defined, and furthermore

$$I = CRT(I_1, I_2, \dots, I_k)$$

is an ideal in $R[x]/(x^n - \lambda)$. Associating a cyclic code with its corresponding ideal we have that

$$CRT(C_1, C_2, \dots, C_k)$$

is λ -constacyclic code over R if and only if each C_i is λ_i -constacyclic code over R_i . ■

Corollary 4.10 *With the above assumptions $R[x]/(x^n - \lambda)$ is principal ring if and only if $R_i[x]/(x^n - \lambda_i)$ is principal ideal for all $1 \leq i \leq k$.*

Proof. Let C a λ -constacyclic code of length n over R , generated by $f(x) \in R[x]/(x^n - \lambda)$ then by Theorem 4.9 and since $C = CRT(C_1, C_2, \dots, C_k)$ then C_i is generated by $\phi_i(f(x))$ which is a polynomial in $R_i[x]/(x^n - \lambda_i)$. So C_i is principal. Conversely, let C_i be a cyclic code of length n over R_i generated by $f_i(x) \in R_i[x]/(x^n - \lambda_i)$. Let $f(x) \in R[x]/(x^n - \lambda)$ such that $f(x) = \phi^{-1}(f_1(x), f_2(x), \dots, f_k(x))$. Since Φ is a ring isomorphism the $f(x)$ is unique. Let D the cyclic code generated by $f(x)$ then

$$D = CRT(C_1, C_2, \dots, C_k)$$

Or the Chinese Remainder Theorem $CRT(C_1, C_2, \dots, C_k)$ is unique thus $C = D$ ■

Remark 4.11 Let R be a finite principal ideal ring, $\prod_{i=1}^k R_i$ its direct decomposition, \mathbb{F}_{q_i} the residue field of each R_i such that $(n, q_i) = 1 \forall i \in \{1, 2, \dots, k\}$. and let $\lambda = CRT(\lambda_1, \lambda_2, \dots, \lambda_k)$. By Theorem 4.3 each $R_i[x]/(x^n - \lambda_i)$ is principal ideal ring, thus $R[x]/(x^n - \lambda)$ is principal ideal ring. If there exist $i \in \{1, \dots, k\}$ such that R_i is a field $R_i[x]/(x^n - \lambda_i)$ is a principal ideal ring for all length.

Example 4.12 Let \mathbb{F}_p be the finite field with p a prime elements and $R = \mathbb{F}_p[x]/(v^2 - v) = \mathbb{F}_p + v\mathbb{F}_p$. Since $\langle v \rangle$ and $\langle (1 - v) \rangle$ are the unique ideal maximal of index of stability 1. Then $(v), (1 - v)$ is the direct decomposition of R . Note that any element c of R^n can be expressed as $c = a + vb = v(a + b) + (1 - v)a$ where $a, b \in \mathbb{F}_p^n$. Let:

$$\begin{aligned} \psi : R^n &\longrightarrow \mathbb{F}_p^n \times \mathbb{F}_p^n \\ a + bv &\mapsto \psi(a + bv) = (\psi_1(a + bv), \psi_2(a + bv)) = (a + b, a) \end{aligned} \quad (7)$$

be the canonical R -module isomorphism. and for $i = 1, 2$, let C_i be a code over \mathbb{F}_p of length n and let

$$C = CRT(C_1, C_2) = \Psi^{-1}(C_1 \times C_2) = \{\Psi^{-1}(\mathbf{v}_1, \mathbf{v}_2) \mid \mathbf{v}_1 \in C_1, \mathbf{v}_2 \in C_2\}.$$

We refer to C as the Chinese product of codes C_1, C_2 [21]. By Theorem 4.9 a λ -constacyclic code over R if and only if each C_i is λ_i -constacyclic code over \mathbb{F}_p . with $\lambda = CRT(\lambda_1, \lambda_2)$. Let $\lambda = 1 - 2v = -v + (1 - v)$ so $\lambda = CRT(-1, 1)$ By Theorem 4.9 any $(1 - 2v)$ -constacyclic code C over R is the Chinese Remainder Theorem of a negacyclic code C_1 over \mathbb{F}_p and a cyclic code C_2 over \mathbb{F}_p such that $C = CRT(C_1, C_2)$.

These codes have also been studied by [37].

Let n be an integer and λ_i, δ_i units in R_i such that $\lambda_i = \delta_i^n$. Let:
 $\mu_i : R_i[x]/\langle x^n - 1 \rangle \mapsto R_i[x]/\langle x^n - \lambda \rangle$ defined by $\mu_i(c(x)) = c(\delta_i^{-1}x)$.
 By Proposition 4.4 we have that μ_i is a ring isomorphism.

In the following we generalize the result above to finite principal ideals rings.

Proposition 4.13 Let n be an integer and $\lambda = CRT(\lambda_1, \dots, \lambda_k)$ and $\delta = CRT(\delta_1, \dots, \delta_k)$ units such that $\lambda = \delta^n$. Let

$$\begin{aligned} \mu : R[x]/\langle x^n - 1 \rangle &\longrightarrow R[x]/\langle x^n - \lambda \rangle \\ c(x) &\mapsto \mu(c(x)) = (\mu_1(c(x)), \dots, \mu_k(c(x))) = (c(\delta_1^{-1}x), \dots, c(\delta_k^{-1}x)). \end{aligned} \quad (8)$$

Then μ is a ring isomorphism.

Proof. We can proof easily that μ is a ring isomorphism, since $R[x]/\langle x^n - 1 \rangle \simeq \prod_{i=1}^k R_i[x]/\langle x^n - 1 \rangle$ and by lemma 4.8 we deduce that $\lambda = \delta^n \iff \lambda_i = \delta_i^n, \forall i \in \{1, \dots, k\}$ then by Proposition 4.4 $\prod_{i=1}^k R_i[x]/\langle x^n - 1 \rangle \simeq \prod_{i=1}^k R_i[x]/\langle x^n - \lambda_i \rangle, \forall i \in \{1, \dots, k\}$ and $R[x]/\langle x^n - \lambda \rangle \simeq \prod_{i=1}^k R_i[x]/\langle x^n - \lambda_i \rangle$ then we obtain the result ■

Since $(n, q_i) = 1$ with \mathbb{F}_{q_i} the residue field of the finite chain ring R_i , then $R_i[x]/\langle x^n - \lambda_i \rangle$ is principal ring $\forall i \in \{1, \dots, k\}$ then the ideals in $R[x]/\langle x^n - \lambda \rangle$ are principal ideals. So the following result is a straightforward consequence of Proposition 4.13.

Corollary 4.14 *Let R be a finite principal ideal ring and λ, δ units in R such that $\lambda = \delta^n$. A subset I in $R[x]$ is an ideal in $R[x]/\langle x^n - 1 \rangle$ if and only if $\mu(I)$ is an ideal in $R[x]/\langle x^n - \lambda \rangle$. Equivalently, the set C is a cyclic code of length n over the principal ideal ring R if and only if $\mu(C)$ is a λ -constacyclic code of length n over R .*

Example 4.15 *Let $R = \mathbb{F}_p[x]/(v^2 - v) \simeq \mathbb{F}_p + v\mathbb{F}_p$, n an odd integer we deduce by 4.13 that any $(1 - 2v)$ -constacyclic code over R is isomorphic to a cyclic code over R . Thus C_1 and C_2 are cyclic codes over \mathbb{F}_p . Conversely if n is odd integer then any negacyclic code C_1 over \mathbb{F}_p is equivalent to cyclic code over \mathbb{F}_p and so by Theorem 4.9 $\text{CRT}(C_1, C_2)$ is cyclic code over R .*

5 Self-dual Cyclic Codes over Finite Ideal Principal Rings

Since any finite ideal principal ring is a direct product of some finite chain rings, one starts by giving some results on the latter.

5.1 Cyclic Self-dual Codes over Finite Chain Rings

In this subsection, we consider cyclic self-dual codes over finite chain rings. For a polynomial $f(x)$ of degree r , let $f^*(x)$ denote its reciprocal polynomial $x^r f(x^{-1})$. The following lemma is easy to obtain.

Lemma 5.1 *Let $f(x)$ and $g(x)$ be two polynomials in $R[x]$ with $\deg f(x) \geq \deg g(x)$. Then the following holds.*

- (i) $[f(x) + g(x)]^* = f(x)^* + x^{\deg f - \deg g} g(x)^*$.
- (ii) If f is monic, then $\overline{f^*} = \overline{f}^*$.

The following theorem gives the structure of the dual of a cyclic code over a finite chain ring.

Theorem 5.2 (*[12, Theorem 3.8]*) *Let R be a finite chain ring with characteristic p , maximal ideal γ , and index of nilpotency e . Let n be an integer such that $(p, n) = 1$ and $f_1 f_2 \dots f_l$ be the representation of $x^n - 1$ as a product of basic irreducible pairwise coprime polynomials in $R[x]$. If C is a cyclic code of length n over R , then $C^\perp = \langle \hat{F}_0^*, \gamma \hat{F}_e^*, \dots, \gamma^{e-1} \hat{F}_2^* \rangle$, where F_0, F_1, \dots, F_{e-1} are pairwise coprime polynomials which are divisors of $x^n - 1$ as given in Theorem 4.3.*

Proposition 5.3 ([12, Proposition 4.3]) *Let R be a finite chain ring with even index of nilpotency e and maximal ideal γ . Then there exists a non-trivial self-dual cyclic code over R if and only if there exists a basic irreducible factor $f \in R[x]$ of $x^n - 1$ such that f and f^* are not associate.*

The following Theorem was given first by Kanwar and López-Permouth [27] and after by H. Dinh and S. R. López-Permouth [12] but with a false proof, so we give it again with another proof. In [12, 27] the authors prove that all cyclotomic coset mod n must be reversible for having the congruence $(p^r)^i \equiv -1 \pmod{n}$ for a positive integers i . While we need only to have the first cyclotomic coset C_1 to be reversible. But before that we give this useful lemma.

Lemma 5.4

$$C_1 \text{ is reversible} \implies \forall j \in \mathbb{Z}_n, C_j \text{ is reversible}$$

Proof. If C_1 is reversible, then there exist k , $1 \leq k \leq \text{ord}_n(q)$ such that $q^k \equiv -1 \pmod{n}$, which means that $jq^k \equiv -j \pmod{n}$, then $C_j = C_{-j}$. \blacksquare

Theorem 5.5 *Let R be a finite chain ring with maximal ideal γ , index of nilpotency e even, and residue field K where $|R| = p^{er}$ and $|K| = p^r$. Then non-trivial cyclic self-dual codes of length n over R exist if and only if $(p^r)^i \not\equiv -1 \pmod{n}$ for all positive integers i .*

Proof. Let $f(x)$ be a monic basic irreducible polynomial which divides $x^n - 1$. Then $\overline{f(x)}$ is a minimal irreducible polynomial over $F_q[x]$. Hence there exists a cyclotomic class C_u associated with $\overline{f(x)}$. Therefore $\overline{f(x)} = \prod_{i \in C_u} (x - \alpha^i)$, where α is a primitive n th root of unity. The reciprocal polynomial of $\overline{f(x)}$ is the polynomial $\overline{f(x)}^* = (\prod_{i \in C_u} (x - \alpha^i))^* = x^r \prod_{i \in C_u} (x^{-1} - \alpha^i) = \prod_{i \in C_{n-u}} (x - \alpha^i)$, but by Lemma 5.1 we have $\overline{f(x)}^* = \overline{f(x)}$.

By Theorem 5.3, a non-trivial cyclic self-dual code exists if and only if there is a basic irreducible polynomial $f(x)$ a factor of $x^n - 1$ such that $f(x)$ and $f(x)^*$ are not associated. We show that this can occur if and only if $(p^r)^i \not\equiv -1 \pmod{n}$ for all positive integers i .

Let $\bar{f}(x) \in F_q[x]$ be irreducible and $f(x)/x^n - 1$. Then $\bar{f}(x) = \prod_{i \in C_u} (x - \alpha^i)$ where C_u is the cyclotomic coset of n that contains the smallest element u and α is a primitive n -th root of unity. Now if $(p^r)^i \not\equiv -1 \pmod{n}$ for all positive integers i , then $C_1 \neq C_{-1}$. Hence $(f(x)) \neq (f^*(x))$ where $\bar{f}(x) = \prod_{i \in C_1} (x - \alpha^i)$, and the code $(fg, \gamma^{\frac{e}{2}} f f^*)$ is a non-trivial self-dual code where $f f^* g = x^n - 1$. Conversely, if a non-trivial cyclic self-dual code exists then by 5.3 there exists a factor $f(x)/x^n - 1$ with $(f(x)) \neq (f(x)^*)$. Hence $C_u \neq C_{-u}$, and then by Lemma 5.4 $C_1 \neq C_{-1}$ where $\bar{f}(x) = \prod_{i \in C_u} (x - \alpha^i)$. Therefore $(p^r)^i \not\equiv -1 \pmod{n}$ for all positive integers i , because otherwise $C_u = C_{-u}$ for all cyclotomic cosets and $(f(x)) = (f(x)^*)$ for any $f(x)/x^n - 1$. \blacksquare

Lemma 5.6 *Let n and s be positive integers, and q a prime power. Then the following holds.*

- (i) *If $q^s \equiv -1 \pmod{n}$, then $\text{ord}_n(q)$ is even.*
- (ii) *If n is prime, then we have $\text{ord}_n(q)$ is even if and only if $\exists i$ such that $q^i \equiv -1 \pmod{n}$.*

Proof. Part (i) follows from [35, Proposition 4.7.5]. For Part (ii), assume that $\text{ord}_n(q) = 2w$ is even, so then $q^{2w} \equiv 1 \pmod{n}$. Hence $n \mid (q^w - 1)(q^w + 1)$. Since n is prime and cannot divide $q^w - 1$ (because of the order), we have that $q^w \equiv -1 \pmod{n}$. The converse follows from Part (i). ■

The following result answers the question posed in [12, p. 1734] by providing a simple criteria for the existence of cyclic self-dual codes.

Theorem 5.7 *Let R be a finite chain ring with maximal ideal γ , index of nilpotency e even, and $|R| = p^{er}$, where $|K| = p^r$. Then non-trivial cyclic self-dual codes of odd length n a power of a prime exist over R if and only if $\text{ord}_n(p^r)$ is odd.*

Proof. If there are no non-trivial self-dual codes, then by Theorem 5.5 there exists an integer i such that $(p^r)^i \equiv -1 \pmod{n}$. Then by Part (i) of Lemma 5.6, we have that $\text{ord}_n(p^r)$ is even.

Conversely, assume that there exists a non-trivial cyclic self-dual code. Then from Theorem 5.5 there is no integer i such that $p^{ri} \equiv -1 \pmod{n}$. We need to show that in this case $\text{ord}_n(p^r)$ is odd. For this, consider the following cases.

- (i) If n is an odd prime, then by Part (ii) of Lemma 5.6, we have $\text{ord}_n(p^r)$ is odd.
- (ii) For $n = q^\alpha$, assume that $\text{ord}_{q^\alpha}(p^r)$ is even. We first must prove the implication

$$\text{ord}_{q^\alpha}(p^r) \text{ is even} \Rightarrow \text{ord}_q(p^r) \text{ is even.}$$

Assume $\text{ord}_{q^\alpha}(p^r)$ is even and $\text{ord}_q(p^r)$ is odd. Then there exist $i > 0$ odd such that $p^{ri} \equiv 1 \pmod{q} \Leftrightarrow p^{ri} = 1 + kq$. Hence $p^{riq^{\alpha-1}} = (1 + kq)^{q^{\alpha-1}} \equiv 1 \pmod{q^\alpha}$, because $(1 + kq)^{q^{\alpha-1}} \equiv 1 + kq^\alpha \pmod{q^{(\alpha+1)}}$ (the proof of the last equality can be found in [11, Lemma 3.30]). Hence

$$p^{riq^{\alpha-1}} \equiv 1 \pmod{q^\alpha}. \tag{9}$$

If we have i odd and $q^{\alpha-1}$ odd, then $\text{ord}_{q^\alpha}(p^r)$ is odd (because $\text{ord}_{q^\alpha}(p^r) \mid iq^{\alpha-1}$), which is absurd. Hence $\text{ord}_q(p^r)$ is even, so there exists some integer j such that $0 < j < \text{ord}_q(p^r)$, and $p^{rj} \equiv -1 \pmod{q}$. Then from (9) we have $p^{rjq^{\alpha-1}} \equiv -1 \pmod{q^\alpha}$. This gives that the cyclotomic class C_1 is reversible, which by Theorem 5.5 is impossible.

■

Remark 5.8 *Note that $\text{ord}_n(p^r)$ odd is a sufficient condition for all n for the existence of a self-dual code over R .*

For the remainder of the paper, the notation $q = \square \pmod n$ means that q is a residue quadratic modulo n .

Corollary 5.9 *Let R be a finite chain ring with maximal ideal γ , index of nilpotency e even, and residue field K such that $|K| = p^r$. Then if $p_1 \dots p_s$ is the prime factorization of an odd integer n such that $p^r = \square \pmod{p_i}$ and $p_i \equiv -1 \pmod 4$ for $1 \leq i \leq s$, then there exists a non-trivial cyclic self-dual code over R .*

Proof. We have that $\text{ord}_n(p^r) = \text{lcm}(\text{ord}_{p_i}(p^r))$. Since $p^r = \square \pmod{p_i}$, then $\text{ord}_{p_i}(p^r)$ divides $\frac{p_i-1}{2}$. Hence $\text{ord}_{p_i}(p^r)$ is odd, otherwise $p_i \equiv 1 \pmod 4$. Then $\text{ord}_n(p^r)$ is odd, and by Theorem 5.7 we have the existence of a non-trivial cyclic self-dual code. ■

Corollary 5.10 *With the previous notation, if n is an odd prime such that $n \equiv -1 \pmod 4$, then there exists a cyclic self-dual code if and only if $p = \square \pmod n$.*

Proof. The necessary condition is given by [12, Corollary 4.7]. For the converse, if we assume $p = \square \pmod n$, then $p^r = \square \pmod n$, and the result follows from Corollary 5.9. ■

For a cyclic code of length n with $(n, p) = 1$, using Theorems 4.2 and 4.3 and Hensel's Lemma, we have the following result.

Theorem 5.11 ([24, Theorem 4.20]) *Let C be a cyclic code of length n over a finite chain ring R with characteristic p such that $(p, n) = 1$. Then C is a free cyclic code with rank k if and only if there is a polynomial $f(x) \in R[x]$ such that $f(x)|(x^n - 1)$ generates C . In this case, we have $k = n - \deg(f)$.*

Theorem 5.12 [3] *Let R be a finite chain ring with maximal ideal $\langle \gamma \rangle$, index of nilpotency e , and characteristic p . Then if p is odd, there is no free cyclic self-dual code of length n over R with $(p, n) = 1$.*

5.2 Self-dual Cyclic Codes over Principal Rings

If R is a finite principal ideal ring, we say that the decomposition of R into a product of finite chain rings, as in (ii), is a *canonical decomposition* of R . The ideal $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$ in this case is called a *direct decomposition* of R .

Let R be a finite ring and $(\mathfrak{a}_i)_{i=1}^n$ a direct decomposition of R . Let $\Psi : R^n \rightarrow \prod_{i=1}^k R_i^n$ be the canonical R -module isomorphism. For $i = 1, \dots, k$, let C_i be a code over R_i of length n and let

$$C = CRT(C_1, C_2, \dots, C_k) = \Psi^{-1}(C_1 \times \dots \times C_k) = \{\Psi^{-1}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) \mid \mathbf{v}_i \in C_i\}.$$

We refer to C as the *Chinese product of codes* C_1, C_2, \dots, C_k [21].

Theorem 5.13 *With the above notation, let C_1, C_2, \dots, C_k be codes of length n with C_i a code over R_i , and let $C = CRT(C_1, C_2, \dots, C_k)$. Then we have the following.*

- (i) C is a cyclic code if and only if each C_i is a cyclic code
- (ii) C_1, C_2, \dots, C_k are self-dual codes if and only if C is a self-dual code.

Proof. The part (i) is a particular case of Theorem 4.9.

$$CRT(C_1, C_2, \dots, C_k)^\perp = CRT(C_1^\perp, C_2^\perp, \dots, C_k^\perp).$$

Then if $C = CRT(C_1, C_2, \dots, C_k)$ we have that

$$C^\perp = CRT(C_1^\perp, C_2^\perp, \dots, C_k^\perp) = CRT(C_1, C_2, \dots, C_k) = C,$$

and the code C is self-dual. ■

In the following we generalize the theorem 5.7 to finite principal ideal rings.

Theorem 5.14 *Let $R \cong \prod_{i=1}^k R/\mathfrak{m}_i^{t_i} = \prod_{i=1}^k R_i$, be a finite principal ideal ring, \mathbb{F}_{q_i} the residue field of R_i for $1 \leq i \leq k$ and C a cyclic code over R . Then C is self-dual code of length a power of a prime odd n if and only if $\text{ord}_n(q_i)$ is odd for $1 \leq i \leq k$.*

Proof. Let n a power of a prime odd such that $(n, q_i) = 1$ and $C = CRT(C_1, C_2, \dots, C_k)$ a cyclic self-dual code over R then by Theorem 5.12 C_i is a cyclic self-dual code over R_i for all $1 \leq i \leq k$ and by Theorem 5.12 $\text{ord}_n(q_i)$ is odd.

Conversely if $\text{ord}_n(q_i)$ is odd then there exist a cyclic self-dual code C_i over R_i for all $1 \leq i \leq k$, then by Theorem 4.9 the cyclic code $C = CRT(C_1, C_2, \dots, C_k)$ is self-dual cyclic code over R . ■

For the remainder of the paper, the notation $q = \square \pmod n$ means that q is a residue quadratic modulo n . In the following we generalize the corollary 5.9 to finite principal ideal rings.

Corollary 5.15 *Let $R \cong \prod_{i=1}^k R/\mathfrak{m}_i^{t_i} = \prod_{i=1}^k R_i$, be a finite principal ideal ring, \mathbb{F}_{q_i} the residue field of R_i , n an integer such that $(n, q - i) = 1$ for $1 \leq i \leq k$. Then if $p_1 \dots p_s$ is the prime factorization of an odd integer n such that $q_i \equiv \square \pmod{p_j}$ and $p_j \equiv -1 \pmod{4}$ for $1 \leq j \leq s$, then there exists a non-trivial cyclic self-dual code over R*

Proof. If $n = p_1 \dots p_s$ such that $q_i \equiv \square \pmod{p_j}$ and $p_j \equiv -1 \pmod{4}$ for $1 \leq j \leq s$. By corollary 5.9 there exist exists a non-trivial cyclic self-dual code C_i over R_i . Then by Theorem 4.9 the cyclic code $C = CRT(C_1, C_2, \dots, C_k)$ is self-dual cyclic code over R . ■

In the following we generalize the corollary 5.10 to finite principal ideal rings.

Corollary 5.16 *With the previous notation, if n is an odd prime such that $n \equiv -1 \pmod{4}$, then there exists a cyclic self-dual code if and only if $p_j \equiv \square \pmod{n}$, where $q_j = p_j^r$.*

Proof. Let n is an odd prime such that $n \equiv -1 \pmod{4}$. If $p_j \equiv \square \pmod{n}$ then by corollary 5.10 there exist a self-dual cyclic code C_j of length n over R_j . Then by Theorem 4.9 the cyclic code $C = CRT(C_1, C_2, \dots, C_k)$ is self-dual cyclic code over R . ■

In the following we generalize the theorem 5.12 to finite principal ideal rings.

Theorem 5.17 *Let $R \cong \prod_{i=1}^k R/\mathfrak{m}_i^{t_i} = \prod_{i=1}^k R_i$, be a finite principal ideal ring, \mathbb{F}_{q_i} the residue field of R_i , n an integer such that $(n, q - i) = 1$ for $1 \leq i \leq k$ and $C = CRT(C_1, C_2, \dots, C_k)$ a cyclic code over R . If there exist $i \in \{1, \dots, k\}$ such that q_i is odd and C_i is free then C is not self-dual.*

Proof. Let $C = CRT(C_1, C_2, \dots, C_k)$ a cyclic code of length n over R such that $(n, q_i) = 1$ for $1 \leq i \leq k$ then By theorem 5.12 if q_i is odd and C_i is free then C_i can not be self-dual, so by Theorem 4.9 C can not be self-dual cyclic code of length n over R . ■

6 Cyclic Codes over Finite Ideal Principal Rings with Odd Index of Stability

In this section, we prove that there is no simple root cyclic self-dual codes over finite chain rings when the nilpotency index of the generator of the maximal ideal is odd and we generalize it to finite ideal principal rings when the stability index of the generator of one of the maximal ideals is odd.

Theorem 6.1 *Let R be a finite chain ring where $\langle \gamma \rangle$ is the maximal ideal with nilpotency index e . If e is odd and q a prime power then there are no nontrivial self-dual cyclic code of length n over R such that $(n, q) = 1$.*

Proof. If $q = 2^k$, then $(n, q) = 1$ and n must be odd, so that from Remark 3.7 e must be even. Let C be a non-trivial cyclic code of length n over R so there exists monic and coprime polynomials $F_0, F_1, \dots, F_{e-1}, F_e$ such that $x^n - 1 = F_0 F_1 \dots F_{e-1} F_e$ and $C = \langle \hat{F}_1, \gamma \hat{F}_2, \dots, \gamma^{e-1} \hat{F}_e \rangle$. If C is self-dual, then from [12, Proposition 4.1] F_i is associate with F_j for $i, j \in \{0, 1, \dots, e\}$ and $i + j \equiv 1 \pmod{e+1}$. Then $F_i = \epsilon F_j^*$ for all $i, j \in \{0, \dots, e\}$ $i + j \equiv 1 \pmod{e+1}$, ϵ a unit in R . Then $F_i \neq F_j^*$ since e is odd and it cannot be that $i + i \equiv e + 2$, so therefore

$$x^n - 1 = F_0 F_0^* F_2 F_2^* F_3 F_3^* \dots F_{\frac{e+1}{2}} F_{\frac{e+1}{2}}^*.$$

Thus none of the F_i are self-reciprocal. The polynomial $(x - 1)$ is a factor of $x^n - 1$, so there is an $0 \leq i_0 \leq e$ such that $F_{i_0} = (x - 1)g(x)$ for some polynomial $g(x)$. Hence

$$F_{i_0}^* = (x - 1)^* g(x)^* = (x - 1)g(x)^* = F_{1-i_0 \pmod{1+e}},$$

which is impossible since for all $0 \leq i \leq e$ the F_i are coprime, and $x^n - 1$ has no repeated roots since $(n, q) = 1$. ■

Theorem 6.2 *Let $R \cong \prod_{i=1}^k R/\mathfrak{m}_i^{t_i}$, be a finite principal ideal ring, and C a cyclic code over R . Then if one of the t_i is odd, C cannot be a self-dual code.*

Proof. From Theorem 4.9, C is cyclic and self-dual if and only if all C_i are also cyclic and self-dual. However, from Theorem 6.1 if there exists an i such that t_i is odd, then the code C_i cannot be self-dual. ■

7 Constacyclic Codes over $R + vR$

Let R be a finite commutative chain ring where $\langle \gamma \rangle$ is the maximal ideal with nilpotency index e and residue field \mathbb{F}_q . Let $R + vR = \{a + vb; a, b \in R\}$ with $v^2 = v$. this ring is a kind of finite commutative principal ideal ring. With two coprime ideals. $\langle v \rangle = \{av; a \in R\}$ and $\langle 1 - v \rangle = \{a(1 - v); a \in R\}$. with index of stability 1 then, both $R_1 = R/\langle v \rangle$ and $R_2 = R/\langle 1 - v \rangle$ is isomorphic to R . By the Chinese Remainder Theorem, we have $R + vR \simeq R_1 \times R_2 \simeq \langle v \rangle \oplus \langle 1 - v \rangle$. The motivation for what we have choused this ring is that the element v and $1 - v$ are nilpotent element such that $v + 1 - v = 1$ so By Proposition 3.6 any submodule N of a module M over $R + vR$ is a direct decomposition of $N_1 \oplus N_2$ where $N_1 = vN$ and $N_2 = (1 - v)N$. In particular for a positive integer n , $(R + vR)^n = v(R + vR)^n \oplus (1 - v)(R + vR)^n$. Since $R + vR \simeq \langle v \rangle \oplus \langle 1 - v \rangle$, let $x_i \in R + vR$ such that $x_i = a_i v + b_i(1 - v)$, $a_i, b_i \in R$ then $x = (x_1, x_2, \dots, x_n) = (a_1 v + b_1(1 - v), a_2 v + b_2(1 - v), \dots, a_n v + b_n(1 - v)) \in (R + vR)^n$ then $x = v(a_1, a_2, \dots, a_n) + (1 - v)(b_1, b_2, \dots, b_n) \in vR^n \oplus (1 - v)R^n$ so $(R + vR)^n = vR^n \oplus (1 - v)R^n$.

Let C a code of length n over $R + vR$ since C is a submodule of $(R + vR)^n$ over $R + vR$ such that

$$C = CRT(C_1, C_2) = \Psi^{-1}(C_1, C_2) = \{\Psi^{-1}(\mathbf{v}_1, \mathbf{v}_2) \mid \mathbf{v}_1 \in C_1, \mathbf{v}_2 \in C_2\}.$$

where C_1 and C_2 are codes of length n over R since the idempotent v and $1 - v$ satisfies $1 + 1 - v = 1$ then by Proposition 3.6 $C = vC + (1 - v)C \simeq C_1 \times C_2$. We use the same proof as for $(R + vR)^n = vR^n \oplus (1 - v)R^n$ for having $vC \simeq vC_1$ and $(1 - v)C \simeq (1 - v)C_2$.

Theorem 7.1 *Let $\lambda = CRT(\lambda_1, \lambda_2) = \lambda_1 v + \lambda_2(1 - v)$ a unit in $R + vR$ such that λ_1, λ_2 are units in R . Let C be a linear code of length n an integer over $R + vR$, Then C is a λ -constacyclic code over $R + vR$ if and only if C_1 and C_2 are a λ_1 -constacyclic code and a λ_2 -constacyclic code respectively over R of length n*

Proof. It is a particular case of Theorem 4.9. ■

Example 7.2 *Let $\lambda = 1 - 2v = -v + (1 - v)$ so $\lambda = CRT(-1, 1)$ By Theorem 4.9 any $(1 - 2v)$ -constacyclic code C over $R + vR$ is the Chinese Remainder Theorem of a negacyclic code C_1 over R and a cyclic code C_2 over R such that $C = CRT(C_1, C_2)$.*

These codes have also been studied by [28].

In [16] Dougherty and al gave the structure of the generator of a cyclic code of length n over \mathbb{Z}_m in a particular case. In the following we give the structure of the generator of a constacyclic code over $R + vR$ in case when the later is principal ideal.

Theorem 7.3 *Let R be a finite commutative chain ring where $\langle \gamma \rangle$ is the maximal ideal with nilpotency index e and residue field \mathbb{F}_q , n a positive integer such that $(n, q) = 1$ (If R is a field we don't need this condition) $\lambda = \lambda_1 v + \lambda_2(1 - v)$ a unit in $R + vR$ such that λ_1, λ_2 are units in R . Let $C = CRT(C_1, C_2)$ be a λ -constacyclic code of length n over $R + vR$, then there are polynomials $f_1(x), f_2(x) \in R[x]$ such that $C = \langle v f_1(x), (1 - v) f_2(x) \rangle$, where $C_1 = \langle f_1(x) \rangle \subseteq R[x]/(x^n - \lambda_1)$ and $C_2 = \langle f_2(x) \rangle \subseteq R[x]/(x^n - \lambda_2)$.*

Proof. By Theorem 4.3 and since $(n, q) = 1$ then the rings $R[x]/(x^n - \lambda_1)$, $R[x]/(x^n - \lambda_2)$ are both principal ideal rings, so there exist polynomials $f_1(x), f_2(x) \in R[x]$ such that $C_1 = \langle f_1(x) \rangle \subseteq R[x]/(x^n - \lambda_1)$ and $C_2 = \langle f_2(x) \rangle \subseteq R[x]/(x^n - \lambda_2)$. For any $c(x) \in C$ there exist polynomials $c_1(x), c_2(x) \in R[x]$ such that $c(x) = v c_1(x) + (1 - v) c_2(x)$ then $c_1(x) \in C_1$, $c_2(x) \in C_2$, there are polynomials $k_1(x), k_2(x) \in R[x]$ such that

$$\begin{aligned} c_1(x) &= k_1(x) f_1(x) \pmod{x^n - \lambda_1} \\ c_2(x) &= k_2(x) f_2(x) \pmod{x^n - \lambda_2} \end{aligned}$$

that means, there are $r_1(x), r_2(x) \in R[x]$ such that $c_1(x) = k_1(x)f_1(x) + r_1(x)(x^n - \lambda_1)$ and $c_2(x) = k_2(x)f_2(x) + r_2(x)(x^n - \lambda_2)$. Since $v(x^n - \lambda) = v(x^n - \lambda_1)$ and $(1 - v)(x^n - \lambda) = (1 - v)(x^n - \lambda_2)$ then

$$\begin{aligned} c(x) &= vc_1(x) + (1 - v)c_2(x) \\ &= v(k_1(x)f_1(x) + r_1(x)(x^n - \lambda_1)) + (1 - v)(k_2(x)f_2(x) + r_2(x)(x^n - \lambda_2)) \\ &= vk_1(x)f_1(x) + (1 - v)k_2(x)f_2(x) + (vr_1(x) + (1 - v)r_2(x))(x^n - \lambda) \end{aligned}$$

hence $vk_1(x)f_1(x) + (1 - v)k_2(x)f_2(x) \pmod{x^n - \lambda}$. So $c(x) \in \langle vf_1(x), (1 - v)f_2(x) \rangle \subset (R + vR)/(x^n - \lambda)$. On the other hand, for any $d(x) \in \langle vf_1(x), (1 - v)f_2(x) \rangle \subset (R + vR)/(x^n - \lambda)$, there are polynomials $k_1(x), k_2(x) \in (R + vR)[x]$ such that

$$d(x) = k_1(x)f_1(x)v + k_2(x)f_2(x)(1 - v) \pmod{x^n - \lambda}$$

then there are $r_1(x), r_2(x) \in R[x]$ such that $vk_1(x) = vr_1(x)$ and $(1 - v)k_2(x) = (1 - v)r_2(x)$ and $r(x) = vr_1(x) + (1 - v)r_2(x)$ such that

$$\begin{aligned} d(x) &= vd_1(x) + (1 - v)d_2(x) \\ &= vf_1(x)r_1(x) + (1 - v)f_2(x)r_2(x) + r(x)(x^n - \lambda) \end{aligned}$$

then

$$\begin{aligned} vd_1(x) &= v(f_1(x)r_1(x) + r_1(x)(x^n - \lambda_1)) \\ (1 - v)d_2(x) &= (1 - v)(f_2(x)r_2(x) + r_2(x)(x^n - \lambda_2)) \end{aligned}$$

this means $d_1(x) \in \langle f_1(x) \rangle \subset R[x]/(x^n - \lambda_1)$ and $d_2(x) \in \langle f_2(x) \rangle \subset R[x]/(x^n - \lambda_2)$ hence $d_1(x) \in C_1, d_2(x) \in C_2$ then $d(x) \in C$ so $\langle vf_1(x), (1 - v)f_2(x) \rangle \subset C$ this gives that $C = \langle vf_1(x), (1 - v)f_2(x) \rangle$ ■

Theorem 7.4 *With the above assumptions Let C be a λ -constacyclic over $R + vR$, then there is a polynomial $f(x) \in (R + vR)[x]$ such that $C = \langle f(x) \rangle$.*

Proof. By Theorem 7.3 there are polynomials $f_1(x)$ and $f_2(x)$ over $R + vR$ such that $C = \langle vf_1(x), (1 - v)f_2(x) \rangle$. Let $f(x) = vf_1(x) + (1 - v)f_2(x)$ obviously $\langle f(x) \rangle \subseteq C$. Note that

$$\begin{aligned} vf(x) &= vf_1(x) \\ (1 - v)f(x) &= (1 - v)f_2(x) \end{aligned}$$

then hence $C = \langle f(x) \rangle$.

7.1 Cyclic Codes over $R + vR$

As a particular case of constacyclic codes over $R + vR$ we investigate in this subsection cyclic codes and their duals over $R + vR$. Let $C = CRT(C_1, C_2)$, By Theorem 4.9 C is a

cyclic code of length n over $R + vR$ if and only if C_1 and C_2 are cyclic codes of length n over R and furthermore By 7.3 there are polynomials $f_1(x), f_2(x) \in R[x]$ such that $C = \langle vf_1(x), (1-v)f_2(x) \rangle$, where $C_1 = \langle f_1(x) \rangle \subseteq R[x]/(x^n - 1)$ and $C_2 = \langle f_2(x) \rangle \subseteq R[x]/(x^n - 1)$. And By Theorem 7.4 $C = \langle f(x) \rangle$ where $f(x) = vf_1(x) + (1-v)f_2(x)$.

Theorem 7.5 *Let R be a finite commutative chain ring where $\langle \gamma \rangle$ is the maximal ideal with nilpotency index e and residue field \mathbb{F}_q and let n an integer such that $(n, q) = 1$ (If R is a field we don't need this condition). If $C = CRT(C_1, C_2)$ then $C^\perp = \langle vh_1(x), (1-v)h_2(x) \rangle$ where $C_1^\perp = \langle h_1(x) \rangle$ and $C_2^\perp = \langle h_2(x) \rangle$.*

Proof. We know that if $C = CRT(C_1, C_2)$ then

$$CRT(C_1, C_2)^\perp = CRT(C_1^\perp, C_2^\perp).$$

So since $(n, q) = 1$, the finite ring $R[x]/(x^n - 1)$ is a principal ideal ring and the dual of any cyclic code is cyclic code then there exist polynomials $h_1(x)$ and $h_2(x)$ in $R[x]$ such that $C_1^\perp = \langle h_1(x) \rangle$ and $C_2^\perp = \langle h_2(x) \rangle$ By Theorem 7.3 $C^\perp = \langle vh_1(x), (1-v)h_2(x) \rangle$ and By theorem 7.4 $C^\perp = \langle vh_1(x) + (1-v)h_2(x) \rangle$. ■

8 Conclusions

In this paper, the isomorphism between constacyclic codes and cyclic codes over finite principal ideal rings was established. Necessary and sufficient conditions were given for the existence of cyclic self-dual codes over finite principal ideals rings.

References

- [1] T. Abualrub and R. Oehmke, On the generators of \mathbb{Z}_4 cyclic codes of length 2^e , IEEE Trans. Inform. Theory, 49(9) 2126–2133, Sept. 2003.
- [2] M.F.Atiya and I.G.Macdonald. Introduction to commutative algebra. Addition-Wesley, 1969.
- [3] A. Batoul, K. Guenda, and T. A. Gulliver, “On self-dual cyclic codes over finite chain rings. Des. Codes Cryptography, 70(3):347-358 (2014).
- [4] A. Batoul, K. Guenda, and T. A. Gulliver, Some constacyclic codes over finite chain rings, Submitted.
- [5] E. Bannai, S. T. Dougherty, M. Harada, and M. Oura, Type II codes, even unimodular lattices and invariant rings, IEEE Trans. Inform. Theory, 45(4) 1194–1205, May 1999.

- [6] T. Blackford, Cyclic codes over \mathbb{Z}_4 of oddly even length, Appl. Discr. Math., 128 27–46, 2003.
- [7] T. Blackford, Negacyclic codes over \mathbb{Z}_4 of even length, IEEE. Trans. Inform. Theory, 49(6) 1417–1424, June 2003.
- [8] A. Bonnetcaze, P. Solé, and A. R. Calderbank, Quaternary quadratic residue codes and unimodular lattices, IEEE Trans. Inform. Theory, 41(2) 366–377, Mar. 1995.
- [9] N. Bourbaki, Commutative Algebra, Springer-Verlag, New York, 1989.
- [10] A. R. Calderbank and N. J. A. Sloane, Modular and p -adic cyclic codes, Designs, Codes, Cryptogr., 6(1) 21–35, 1996.
- [11] M. Demazure, Cours D’Algèbre: Primalité, Divisibilité, Codes, Cassini, Paris, 1997.
- [12] H. Dinh and S. R. López-Permouth, Cyclic and negacyclic codes over finite chain rings, IEEE Trans Inform Theory, 50(8) 1728–1744, Aug. 2004.
- [13] S. T. Dougherty, T. A. Gulliver, and J. N. C. Wong, Self-dual codes over \mathbb{Z}_8 and \mathbb{Z}_9 , Des., Codes, Cryptogr., 41(3) 235–249, 2006.
- [14] S.T. Dougherty, H. Liu, and Y.H. Park, Lifted codes over finite chain rings, Math. J. Okayama University, 53 39–53, Jan. 2010.
- [15] S. T. Dougherty and J. L. Kim, Construction of self-dual codes over chain rings, Int. J. Inform. and Coding Theory, 1(2) 171–190 2010.
- [16] S.T. Dougherty, J. L. Kim and H. Kulosman, MDS codes over finite principal ideal rings, Designs, Codes and Cryptography, 50, 77–92, 2009.
- [17] S.T. Dougherty and K. Shiromoto, MDR Codes codes over \mathbb{Z}_k , IEEE Trans. Inform. Theory, vol. 46, no. 1, 2000, 265–269.
- [18] S.T. Dougherty, M. Harada, and P. Solé, Self-dual codes over rings and the Chinese remainder theorem, Hokkaido Math Journal, 28 253–283, 1999.
- [19] S. T. Dougherty, H. Liu, and Y. H. Park, Lifted codes over finite chain rings, Math. J. Okayama Univ., 53 39–53, Jan. 2010.
- [20] S. T. Dougherty, M. Harada, and P. Solé, Self-dual codes over rings and the Chinese remainder theorem, Hokkaido Math. J., 28 253–283, 1999.
- [21] S.T. Dougherty and K. Shiromoto, MDR Codes codes over \mathbb{Z}_k , IEEE Trans. Inform. Theory, 46(1) 265–269, 2000.

- [22] S. T. Dougherty and J. L. Kim, Construction of self-dual codes over chain rings, *Int. J. Inform. and Coding Theory*, 1(2) 171–190 2010.
- [23] K. Guenda, New MDS self-dual codes over finite fields, *Des., Codes, Cryptogr.*, 2012.
- [24] K. Guenda and T. A. Gulliver, MDS and self-dual codes over rings, *Finite Fields Appl.*, 2011.
- [25] A. R. Hammons Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, The \mathbb{Z}_4 linearity of Kerdock, Preparata, Goethals and related codes, *IEEE Trans. Inform. Theory*, 40(2) 301–319, Mar. 1994.
- [26] Y. Jia, S. Ling, and C. Xing, On self-dual cyclic codes over finite fields, *IEEE Trans. Inform. Theory*, 57(4) 2243–2251, Apr. 2011.
- [27] P. Kanwar and S. R. López-Permouth, Cyclic codes over the integers modulo p^m , *Finite Fields Appl.*, 3(4) 334–352, Oct. 1997.
- [28] Dajian Liao, Yuansheng Tang, A Class of Constacyclic Codes over $R + vR$ and Its Gray Image *Int. J. Communications, Network and System Sciences*, 2012, 5, 222–227.
- [29] S. R. López-Permouth and S. Szabo, Repeated root cyclic and negacyclic codes over Galois rings, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, Springer Lecture Notes in Computer Science, 5527 219–222, 2009.
- [30] B. R. McDonald, *Finite rings with identity*, in *Pure and Applied Mathematics*, New-York Marcel Dekker, 28, 1974.
- [31] G. Ganske and B. R. McDonald, *Finite local rings*, *Rocky Mountain J. Math.* 3(4), 521–540, 1973.
- [32] P. Moree and P. Solé, Around Pellikán’s conjecture on very odd sequences, *Manuscripta Math.*, 117(2) 219–238, June 2005.
- [33] G. H. Norton and A. Sălăgean, On the structure of linear and cyclic codes over a finite chain ring, *Appl. Algebra Engr. Comm. Comput.*, 10(6) 489–506, 2000.
- [34] E. Rains and N. J. A. Sloane, Self-dual codes, in *Handbook of Coding Theory*, V.S. Pless and W.C. Huffman, eds., Elsevier, Amsterdam, 177–294, 1998.
- [35] G. Skersys, Calcul du group d’automorphismes des codes, PhD Thesis, Laco, Limoges, 1999.
- [36] J. Wolfmann, Negacyclic and cyclic codes over \mathbb{Z}_4 , *IEEE Trans. Inform. Theory*, 45(7) 2522–2532, Nov. 1999.

- [37] S.X.Zhu and L.Wang, A Class of Constacyclic Codes over $\mathbb{F}_p + v\mathbb{F}_p$ and its Gray Image, Discrete Mathematics, Vol.311, No.9, 2011, pp.2677-2682.